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Hyper- and reverse-Wiener indices of F-sums of graphs

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ABSTRACT

The Wiener index $W(G) = \sum_{\{u,v\}\subset V(G)} d(u, v)$, the hyper-Wiener index $WW(G) = \frac{1}{2}\sum_{\{u,v\}\subset V(G)} \left[d(u, v) + d^2(u, v)\right]$ and the reverse-Wiener index $\Lambda(G) = \frac{n(n-1)D}{2} - W(G)$, where $d(u, v)$ is the distance of two vertices u, v in G, $d^2(u, v) = d(u, v)^2$, $n = |V(G)|$ and D is the diameter of G. In [M. Eliasi, B. Taeri, Four new sums of graphs and their Wiener indices, Discrete Appl. Math. 157 (2009) 794–803], Eliasi and Taeri introduced the F-sums of two connected graphs. In this paper, we determine the hyper- and reverse-Wiener indices of the F-sum graphs and, subject to some condition, we present some exact expressions of the reverse-Wiener indices of the F-sum graphs.

1. Introduction

Configuration of nodes (vertices) and connections (edges) occurs in a great diversity of applications. They also represent organic molecules. Graph theory was successfully provided the chemist with a variety of very useful tools, namely, topological indices. A topological index is a real number related to a graph and it is a structural invariant, i.e., it does not depend on the labeling or the pictorial representation. There are several topological indices have been defined and many of them have found applications as means for modeling chemical, pharmaceutical and other properties of molecules. Among them, the Wiener index W (Wiener number) was introduced in 1947 by Wiener and it is one of the oldest and most thoroughly examined molecular graph-based structural descriptor of organic molecule [24,7,8]. Since Wiener index is applicable to acyclic (tree) graphs only, various graph topological indices involving the Wiener index have been studied [18,1,12,11,15,21,23].

The hyper-Wiener index of acyclic graphs was introduced by Randic in 1993. Then Klein et al. generalized Randic's definition for all connected graphs, as a generalization of the Wiener index [18]. For the mathematical properties of hyper-Wiener index and its applications in chemistry we refer to [3,4,10,17,16,19,26].

The reverse-Wiener index was proposed by Balaban et al. in 2000 [1]; it is important for a reverse problem and it is also found applications in modeling of structure–property relations [1,14]. Some mathematical properties of the reverse-Wiener index may be found in [2,20,25].

2. Definitions

When investigating graph parameters, it is clear that we only need to consider connected graphs. In computing the hyper- and reverse-Wiener indices, we also consider only simple, finite and undirected graphs. Hence, in what follows, we may assume that all graphs are connected, simple, finite and undirected. For terminology and notation not defined here we

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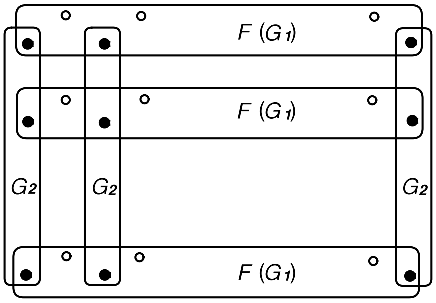

Fig. 1. The F-sum graph $G_1 +_F G_2$ of two connected graphs G_1, G_2.

refer to [6,13,22]. Let $G = (V(G), E(G))$ be a connected graph with the vertex set $V(G)$ and edge set $E(G)$. The symbol $d_G(u, v)$ denotes the distance between the vertices u and v of G defined as the length (number of edges) of a shortest u–v path in G (when there is no ambiguity, we omit the subscript G). The diameter of G is defined as $D(G) = \max\{d(u, v) : u, v \in V(G)\}$.

Since, in this paper, we only consider F-sums of two connected graphs, throughout the paper we set $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and $V_1 = \{u_1, \ldots, u_{n_1}\}$, $V_2 = \{v_1, \ldots, v_{n_2}\}$, $E_1 = \{e_1, \ldots, e_{m_1}\}$ (when we are considering $F(G)$ and F-sums, e_i is also viewed as a new added vertex).

Definition 1 (*[13,5]*)**.** Sum graph (Cartesian product) $G_1 + G_2$:

$$V(G_1 + G_2) = V_1 \times V_2,$$
$$E(G_1 + G_2) = \{(u_i, v_j)(u_s, v_t) : i = s \text{ and } v_j v_t \in E_2 \vee j = t \text{ and } u_i u_s \in E_1\}.$$

Definition 2 (*[5]*)**.** For a connected graph G, define five related graphs as follows:

1. $L(G)$ (*line graph*): The vertices of $L(G)$ are the edges of G. Two edges of G that share a vertex are considered to be adjacent in $L(G)$.
2. $S(G)$ (*subdivision graph*) is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.
3. $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.
4. $Q(G)$ is obtained from G by inserting a new vertex in to each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.
5. $T(G)$ (*total graph*) has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

It is clear from the above definitions that

$$V(L(G)) = E(G),$$
$$V(S(G)) = V(R(G)) = V(Q(G)) = V(T(G)) = V(G) \cup E(G),$$
$$E(R(G)) = E(G) \cup E(S(G)),$$
$$E(Q(G)) = E(L(G)) \cup E(S(G)),$$
$$E(T(G)) = E(G) \cup E(L(G)) \cup E(S(G)), \text{ where } E(G), E(L(G)), E(S(G)) \text{ are mutually disjoint.}$$

Definition 3 (*[9]*)**.** F-sum graph $G_1 +_F G_2$ for $F \in \{S, R, Q, T\}$ define as

$$\begin{aligned} V(G_1 +_F G_2) &= (V_1 \cup E_1) \times V_2, \\ E(G_1 +_F G_2) &= \{b_{ij} b_{st} : i = s \text{ and } v_j v_t \in E_2 \vee j = t \text{ and } u_i u_s \in E(F(G_1))\} \\ &\quad \cup \{w_{ij} w_{sj} : e_i e_s \in E(F(G_1))\} \cup \{w_{ij} b_{sj} : e_i u_s \in E(F(G_1))\}, \end{aligned}$$

where $b_{ij} = (u_i, v_j)$ and $w_{ij} = (e_i, v_j)$ (we refer to b_{ij} as a black vertex and w_{ij} as a white vertex).

Note that $G_1 +_F G_2$ has n_2 copies of the graph $F(G_1)$ and n_1 copies of the graph G_2, moreover, the copies of $F(G_1)$ and the copies of G_2 are edge disjoint and each edge of $G_1 +_F G_2$ belongs to one of the copies of $F(G_1)$ or G_2, see Fig. 1.

Definition 4. Wiener index $W(G)$ ([24])

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u, v).$$

Hyper-Wiener index $WW(G)$ ([18])

$$WW(G) = \frac{1}{2} \sum_{\{u,v\} \subset V(G)} \left[d(u, v) + d^2(u, v) \right],$$

where $d^2(u, v) = d(u, v)^2$;

Reverse-Wiener index $\Lambda(G)$ ([1])

$$\Lambda(G) = \frac{n(n-1)D}{2} - W(G),$$

where $n = |V(G)|$ and D is the diameter of G.

Our goal in this paper is to determine the hyper- and reverse-Wiener indices of F-sum graphs. In Section 2, formulae for computing the hyper-Wiener indices of the F-sum graphs $G_1 +_F G_2$ of two connected graphs are obtained. In Section 3, the reverse-Wiener indices of the F-sum graphs of two connected graphs are computed and, subject to some condition, some exact expressions of the reverse-Wiener indices of the F-sum graphs are presented.

3. Hyper-Wiener indices of F-sum graphs

$d_{G_1+G_2}(b_{ij}, b_{st}) = d_{G_1}(u_i, u_s) + d_{G_2}(v_j, v_t)$ (distance lemma) is a well-known property of Cartesian product graphs. By this result and careful observing, one can obtain the following lemma.

Lemma 1 *([9]). Let G_1 and G_2 be two connected graphs. Then the distance of two vertices in $G_1 +_F G_2$ is the following.*

1. $d_{G_1 +_F G_2}(b_{ij}, b_{st}) = d_{F(G_1)}(u_i, u_s) + d_{G_2}(v_j, v_t)$ *for* $F \in \{S, R, Q, T\}$;
2. $d_{G_1 +_F G_2}(b_{ij}, w_{st}) = d_{F(G_1)}(u_i, e_s) + d_{G_2}(v_j, v_t)$ *for* $F \in \{S, R, Q, T\}$;
3. $d_{G_1 +_F G_2}(w_{ij}, w_{st}) = \begin{cases} 2 + d_{G_2}(v_j, v_t), & \text{if } i = s \\ d_{F(G_1)}(e_i, e_s) + d_{G_2}(v_j, v_t), & \text{if } i \neq s \end{cases}$ *for* $F \in \{S, R\}$;
4. $d_{G_1 +_F G_2}(w_{ij}, w_{st}) = \begin{cases} 2 + d_{G_2}(v_j, v_t), & \text{if } i = s \\ d_{F(G_1)}(e_i, e_s), & \text{if } i \neq s \text{ and } j = t \\ 1 + d_{F(G_1)}(e_i, e_s) + d_{G_2}(v_j, v_t), & \text{if } i \neq s \text{ and } j \neq t \end{cases}$

   for $F \in \{Q, T\}$.

Now we are ready to state a main result of this paper.

Theorem 1. *Let G_i be a connected graph with n_i vertices and m_i edges. Then*

$$\begin{aligned} WW(G_1 +_F G_2) &= n_2^2 WW(F(G_1)) + (n_1 + m_1)^2 WW(G_2) + 2W(F(G_1))W(G_2) + 2m_1 W(G_2) + \frac{3}{2} n_2^2 m_1 \\ &\quad \textit{for } F \in \{S, R\}; \\ WW(G_1 +_F G_2) &= n_2^2 WW(F(G_1)) + (n_1 + m_1)^2 WW(G_2) + 2W(F(G_1))W(G_2) \\ &\quad + (n_2^2 - n_2)W(L(G_1)) + (m_1^2 + m_1)W(G_2) + \frac{3}{2} n_2^2 m_1 + \frac{1}{2}(n_2^2 - n_2)(m_1^2 - m_1) \quad \textit{for } F \in \{Q, T\}. \end{aligned}$$

Proof. We separately consider the distances between black and black vertices, black and white vertices and white and white vertices. Set $O := \{1, \dots, n_1\}$, $P := \{1, \dots, m_1\}$, $Q := \{1, \dots, n_2\}$. By Lemma 1 and the definitions of Wiener and hyper-Wiener indices, we have

$$\begin{aligned} A &:= \sum_{i,s \in O\, j,t \in Q} \left[d_{G_1 +_F G_2}(b_{ij}, b_{st}) + d^2_{G_1 +_F G_2}(b_{ij}, b_{st}) \right] \\ &= \sum_{i,s \in O\, j,t \in Q} \left[d_{F(G_1)}(u_i, u_s) + d^2_{F(G_1)}(u_i, u_s) \right] + \sum_{i,s \in O\, j,t \in Q} \left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t) \right] \\ &\quad + 2 \sum_{i,s \in O\, j,t \in Q} d_{F(G_1)}(u_i, u_s) d_{G_2}(v_j, v_t) \\ &= \sum_{j,t \in Q} \sum_{i,s \in O} \left[d_{F(G_1)}(u_i, u_s) + d^2_{F(G_1)}(u_i, u_s) \right] + \sum_{i,s \in O} \sum_{j,t \in Q} \left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t) \right] \\ &\quad + 2 \sum_{i,s \in O} d_{F(G_1)}(u_i, u_s) \sum_{j,t \in Q} d_{G_2}(v_j, v_t) \\ &= n_2^2 \sum_{i,s \in O} \left[d_{F(G_1)}(u_i, u_s) + d^2_{F(G_1)}(u_i, u_s) \right] + 4n_1^2 WW(G_2) + 4W(G_2) \sum_{i,s \in O} d_{F(G_1)}(u_i, u_s); \end{aligned}$$

$$\begin{aligned}
B &:= \sum_{i\in O}\sum_{s\in P}\sum_{j,t\in Q}\left[d_{G_1+_F G_2}(b_{ij}, w_{st}) + d^2_{G_1+_F G_2}(b_{ij}, w_{st})\right] \\
&= \sum_{j,t\in Q}\sum_{i\in O}\sum_{s\in P}\left[d_{F(G_1)}(u_i, e_s) + d^2_{F(G_1)}(u_i, e_s)\right] + \sum_{i\in O}\sum_{s\in P}\sum_{j,t\in Q}\left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t)\right] \\
&\quad + 2\sum_{i\in O}\sum_{s\in P} d_{F(G_1)}(u_i, e_s)\sum_{j,t\in Q} d_{G_2}(v_j, v_t) \\
&= n_2^2\sum_{i\in O}\sum_{s\in P}\left[d_{F(G_1)}(u_i, e_s) + d^2_{F(G_1)}(u_i, e_s)\right] + 4n_1m_1WW(G_2) + 4W(G_2)\sum_{i\in O}\sum_{s\in P} d_{F(G_1)}(u_i, e_s);
\end{aligned}$$

Case 1. $F \in \{R, S\}$

$$\begin{aligned}
C &:= \sum_{i,s\in P}\sum_{j,t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] \\
&= \sum_{i=s\in P}\sum_{j,t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] + \sum_{i\neq s\in P}\sum_{j,t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] \\
&= \sum_{j,t\in Q}\sum_{i=s\in P} 6 + \sum_{i=s\in P}\sum_{j,t\in Q}\left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t)\right] + \sum_{i=s\in P}\sum_{j,t\in Q} 4d_{G_2}(v_j, v_t) \\
&\quad + \sum_{j,t\in Q}\sum_{i\neq s\in P}\left[d_{F(G_1)}(e_i, e_s) + d^2_{F(G_1)}(e_i, e_s)\right] + \sum_{i\neq s\in P}\sum_{j,t\in Q}\left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t)\right] \\
&\quad + 2\sum_{i\neq s\in P} d_{F(G_1)}(e_i, e_s)\sum_{j,t\in Q} d_{G_2}(v_j, v_t) \\
&= 6n_2^2m_1 + n_2^2\sum_{i,s\in P}\left[d_{F(G_1)}(e_i, e_s) + d^2_{F(G_1)}(e_i, e_s)\right] + 8m_1W(G_2) + 4m_1^2WW(G_2) + 4W(G_2)\sum_{i,s\in P} d_{F(G_1)}(e_i, e_s);
\end{aligned}$$

Case 2. $F \in \{Q, T\}$

$$\begin{aligned}
C &:= \sum_{i,s\in P}\sum_{j,t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] \\
&= \sum_{i=s\in P}\sum_{j,t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] + \sum_{i\neq s\in P}\sum_{j=t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] \\
&\quad + \sum_{i\neq s\in P}\sum_{j\neq t\in Q}\left[d_{G_1+_F G_2}(w_{ij}, w_{st}) + d^2_{G_1+_F G_2}(w_{ij}, w_{st})\right] \\
&= \sum_{j,t\in Q}\sum_{i=s\in P} 6 + \sum_{i=s\in P}\sum_{j,t\in Q}\left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t)\right] + \sum_{i=s\in P}\sum_{j,t\in Q} 4d_{G_2}(v_j, v_t) \\
&\quad + \sum_{j=t\in Q}\sum_{i\neq s\in P}\left[d_{F(G_1)}(e_i, e_s) + d^2_{F(G_1)}(e_i, e_s)\right] + \sum_{j\neq t\in Q}\sum_{i\neq s\in P} 2 + \sum_{j\neq t\in Q}\sum_{i\neq s\in P}\left[d_{F(G_1)}(e_i, e_s) + d^2_{F(G_1)}(e_i, e_s)\right] \\
&\quad + \sum_{i\neq s\in P}\sum_{j\neq t\in Q}\left[d_{G_2}(v_j, v_t) + d^2_{G_2}(v_j, v_t)\right] + 2\sum_{i\neq s\in P} d_{F(G_1)}(e_i, e_s)\sum_{j\neq t\in Q} d_{G_2}(v_j, v_t) + \sum_{j\neq t\in Q}\sum_{i\neq s\in P} 2d_{F(G_1)}(e_i, e_s) \\
&\quad + \sum_{i\neq s\in P}\sum_{j\neq t\in Q} 2d_{G_2}(v_j, v_t) \\
&= 6n_2^2m_1 + n_2^2\sum_{i,s\in P}\left[d_{F(G_1)}(u_i, u_s) + d^2_{F(G_1)}(u_i, u_s)\right] + 4(m_1^2 + m_1)W(G_2) \\
&\quad + 4m_1^2WW(G_2) + 2\big(2W(G_2) + n_2^2 - n_2\big)\sum_{i,s\in P} d_{F(G_1)}(e_i, e_s) + 2(n_2^2 - n_2)(m_1^2 - m_1).
\end{aligned}$$

If $F \in \{R, S\}$, then, again by the definition of hyper-Wiener index, we have

$$\begin{aligned}
WW(G_1 +_F G_2) &= \frac{1}{4}(A + 2B + C) \\
&= \frac{1}{2}n_2^2\left(\sum_{i<s\in O}\left[d_{F(G_1)}(u_i, u_s) + d^2_{F(G_1)}(u_i, u_s)\right] + \sum_{i\in O}\sum_{s\in P}\left[d_{F(G_1)}(u_i, e_s) + d^2_{F(G_1)}(u_i, e_s)\right]\right. \\
&\quad \left. + \sum_{i<s\in P}\left[d_{F(G_1)}(e_i, e_s) + d^2_{F(G_1)}(e_i, e_s)\right]\right) + (n_1 + m_1)^2WW(G_2)
\end{aligned}$$

$$+2W(G_2)\left(\sum_{i<s\in O} d_{F(G_1)}(u_i,u_s)+\sum_{i\in O\ s\in P} d_{F(G_1)}(u_i,e_s)+\sum_{i<s\in P} d_{F(G_1)}(e_i,e_s)\right)$$
$$+\frac{3}{2}n_2^2m_1+2m_1W(G_2)$$
$$=n_2^2WW(F(G_1))+(n_1+m_1)^2WW(G_2)+2W(F(G_1))W(G_2)+2m_1W(G_2)+\frac{3}{2}n_2^2m_1.$$

If $F\in\{Q,T\}$, then we have

$$\begin{aligned}WW(G_1+_F G_2) &= \frac{1}{4}(A+2B+C)\\
&= \frac{1}{2}n_2^2\left(\sum_{i<s\in O}\left[d_{F(G_1)}(u_i,u_s)+d^2_{F(G_1)}(u_i,u_s)\right]+\sum_{i\in O\ s\in P}\left[d_{F(G_1)}(u_i,e_s)+d^2_{F(G_1)}(u_i,e_s)\right]\right.\\
&\quad \left.+\sum_{i<s\in P}\left[d_{F(G_1)}(e_i,e_s)+d^2_{F(G_1)}(e_i,e_s)\right]\right)+(n_1+m_1)^2WW(G_2)\\
&\quad +2W(G_2)\left(\sum_{i<s\in O} d_{F(G_1)}(u_i,u_s)+\sum_{i\in O\ s\in P} d_{F(G_1)}(u_i,e_s)+\sum_{i<s\in P} d_{F(G_1)}(e_i,e_s)\right)\\
&\quad +(m_1^2+m_1)W(G_2)+(n_2^2-n_2)\sum_{i<s\in P} d_{F(G_1)}(e_i,e_s)+\frac{3}{2}n_2^2m_1+\frac{1}{2}(n_2^2-n_2)(m_1^2-m_1)\\
&= n_2^2WW(F(G_1))+(n_1+m_1)^2WW(G_2)+2W(F(G_1))W(G_2)+(n_2^2-n_2)W(L(G_1))\\
&\quad +(m_1^2+m_1)W(G_2)+\frac{3}{2}n_2^2m_1+\frac{1}{2}(n_2^2-n_2)(m_1^2-m_1). \quad \square\end{aligned}$$

4. Reverse-Wiener indices of F-sum graphs

Theorem 2 ([9]). *Let G_i be a connected graph with n_i vertices and m_i edges and $F=S$ or R. Then*

$$W(G_1+_F G_2)=n_2^2W(F(G_1))+(n_1+m_1)^2W(G_2)+m_1(n_2^2-n_2).$$

Theorem 3 ([9]). *Let G_i be a connected graph with n_i vertices and m_i edges and $F=Q$ or T. Then*

$$W(G_1+_F G_2)=n_2^2W(F(G_1))+(n_1+m_1)^2W(G_2)+\frac{1}{2}(n_2^2-n_2)(m_1^2-m_1).$$

Observation 1. *Let G be a connected graph. Then*

1. $D(S(G))=2D(G)$;
2. $D(R(G))=D(G)$;
3. $D(Q(G))=D(G)+1$;
4. $D(T(G))=D(G)$.

By Lemma 1 and Observation 1, we have the following lemma.

Lemma 2. *Let G_i be connected graphs with the diameter D_i. Then*

1. $D(G_1+_S G_2)=2D_1+D_2$;
2. $D(G_1+_R G_2)=\max\{D_1+D_2,2+D_2\}$;
3. $D(G_1+_Q G_2)=D_1+D_2+1$;
4. $D(G_1+_T G_2)=\max\{D_1+D_2,2+D_2\}$;

The following theorem is directly obtained by Theorems 2 and 3 and Lemma 2.

Theorem 4. *Let G_i be a connected graph with diameter D_i ($D_1 > 1$) have n_i vertices and m_i edges. Then*

$$\begin{aligned}
\Lambda(G_1 +_S G_2) &= \frac{1}{2}(2D_1 + D_2)\big[n_2^2(n_1+m_1)^2 - n_2(n_1+m_1)\big] \\
&\quad - n_2^2 W(F(G_1)) - (n_1+m_1)^2 W(G_2) - m_1(n_2^2 - n_2); \\
\Lambda(G_1 +_R G_2) &= \frac{1}{2}(D_1 + D_2)\big[n_2^2(n_1+m_1)^2 - n_2(n_1+m_1)\big] \\
&\quad - n_2^2 W(F(G_1)) - (n_1+m_1)^2 W(G_2) - m_1(n_2^2 - n_2) \quad \text{for } D_1 > 1; \\
\Lambda(G_1 +_Q G_2) &= \frac{1}{2}(D_1 + D_2 + 1)\big[n_2^2(n_1+m_1)^2 - n_2(n_1+m_1)\big] \\
&\quad - n_2^2 W(F(G_1)) - (n_1+m_1)^2 W(G_2) - \frac{1}{2}(n_2^2 - n_2)(m_1^2 - m_1); \\
\Lambda(G_1 +_T G_2) &= \frac{1}{2}(D_1 + D_2)\big[n_2^2(n_1+m_1)^2 - n_2(n_1+m_1)\big] \\
&\quad - n_2^2 W(F(G_1)) - (n_1+m_1)^2 W(G_2) - \frac{1}{2}(n_2^2 - n_2)(m_1^2 - m_1) \quad \text{for } D_1 > 1.
\end{aligned}$$

Obviously, if $D(G) = 1$, then G is a complete graph. In the following, we compute reverse-Wiener indices of F-sums of graphs for G_1 is a complete graph.

Lemma 3. *Let K_n be a complete graph on n vertices. Then*

$$\begin{aligned}
W(S(K_n)) &= \frac{1}{2}n(n-1)\left[3n + \frac{1}{2}n(n-1) + \frac{1}{2}(n-2)(n-3) - 3\right]; \\
W(R(K_n)) &= \frac{1}{2}n(n-1)\left[2(n-1) + \frac{1}{2}n(n-1) + \frac{1}{4}(n-2)(n-3)\right]; \\
W(Q(K_n)) &= \frac{1}{2}n(n-1)\left[2n + \frac{1}{4}\big(n(n-1)+2\big) + \frac{1}{4}(n-2)(n-3) - 1\right]; \\
W(T(K_n)) &= \frac{1}{2}n(n-1)\left[2(n-1) + \frac{1}{4}\big(n(n-1)+2\big) + \frac{1}{4}(n-2)(n-3)\right].
\end{aligned}$$

Proof. As in the definition of F-sum graphs, we refer to the vertices of K_n as black vertices and the new added vertices as white vertices. Then the sum of all distance in $F(G)$ equals the sum of the sums of distances between black and black vertices, black and white vertices and white and white vertices. Set $O := \{1, \dots, n\}$, $P := \left\{1, \dots, \binom{n}{2}\right\}$. Then

$$\begin{aligned}
W(S(K_n)) &= \sum_{i<j\in O} d(v_i, v_j) + \sum_{i\in p}\sum_{j\in O} d(e_i, v_j) + \frac{1}{2}\sum_{i\in P}\sum_{j\in P} d(e_i, e_j) \\
&= \sum_{i<j\in O} 2 + \sum_{i\in p}\big[3(n-2)+2\big] + \frac{1}{2}\sum_{i\in P}\left[2\left(\binom{n}{2}-1\right) + 2\binom{n-2}{2}\right] \\
&= \frac{1}{2}n(n-1)\left[3n + \frac{1}{2}n(n-1) + \frac{1}{2}(n-2)(n-3) - 3\right]; \\
W(R(K_n)) &= \sum_{i<j\in O} d(v_i, v_j) + \sum_{i\in p}\sum_{j\in O} d(e_i, v_j) + \frac{1}{2}\sum_{i\in P}\sum_{j\in P} d(e_i, e_j) \\
&= \sum_{i<j\in O} 1 + \sum_{i\in p} 2(n-1) + \frac{1}{2}\sum_{i\in P}\left[2\left(\binom{n}{2}-1\right) + \binom{n-2}{2}\right] \\
&= \frac{1}{2}n(n-1)\left[2(n-1) + \frac{1}{2}n(n-1) + \frac{1}{4}(n-2)(n-3)\right];
\end{aligned}$$

$$
\begin{aligned}
W(Q(K_n)) &= \sum_{i<j\in O} d(v_i, v_j) + \sum_{i\in p}\sum_{j\in O} d(e_i, v_j) + \frac{1}{2}\sum_{i\in P}\sum_{j\in P} d(e_i, e_j) \\
&= \sum_{i<j\in O} 2 + \sum_{i\in p} 2(n-1) + \frac{1}{2}\sum_{i\in P}\left[\binom{n}{2} - 1 + \binom{n-2}{2}\right] \\
&= \frac{1}{2}n(n-1)\left[2n + \frac{1}{4}\big(n(n-1)+2\big) + \frac{1}{4}(n-2)(n-3) - 1\right];
\end{aligned}
$$

$$
\begin{aligned}
W(T(K_n)) &= \sum_{i<j\in O} d(v_i, v_j) + \sum_{i\in p}\sum_{j\in O} d(e_i, v_j) + \frac{1}{2}\sum_{i\in P}\sum_{j\in P} d(e_i, e_j) \\
&= \sum_{i<j\in O} 1 + \sum_{i\in p} 2(n-1) + \frac{1}{2}\sum_{i\in P}\left[\binom{n}{2} - 1 + \binom{n-2}{2}\right] \\
&= \frac{1}{2}n(n-1)\left[2(n-1) + \frac{1}{4}\big(n(n-1)+2\big) + \frac{1}{4}(n-2)(n-3)\right]. \quad \square
\end{aligned}
$$

By Theorems 2 and 3 and Lemma 3, we have

Theorem 5. *Let G be a connected graph on s vertices and with diameter D. Then*

$$
\begin{aligned}
\Lambda(K_n +_S G) &= \frac{1}{2}(2+D)\left[\left(\frac{1}{2}n(n+1)s\right)^2 - \frac{1}{2}n(n+1)s\right] - \frac{1}{4}n^2(n+1)^2 W(G) \\
&\quad - \frac{1}{2}n(n-1)s^2\left[3n + \frac{1}{2}n(n-1) + \frac{1}{2}(n-2)(n-3) - \frac{1}{s} - 2\right]; \\
\Lambda(K_n +_R G) &= \frac{1}{2}(2+D)\left[\left(\frac{1}{2}n(n+1)s\right)^2 - \frac{1}{2}n(n+1)s\right] - \frac{1}{4}n^2(n+1)^2 W(G) \\
&\quad - \frac{1}{2}n(n-1)s^2\left[2n + \frac{1}{2}n(n-1) + \frac{1}{4}(n-2)(n-3) - \frac{1}{s} - 1\right]; \\
\Lambda(K_n +_Q G) &= \frac{1}{2}(2+D)\left[\left(\frac{1}{2}n(n+1)s\right)^2 - \frac{1}{2}n(n+1)s\right] - \frac{1}{4}n^2(n+1)^2 W(G) \\
&\quad - \frac{1}{2}n(n-1)s^2\left[2n + \frac{1}{4}\big(n(n-1)+2\big) + \frac{1}{4}(n-2)(n-3) - 1\right] \\
&\quad - \frac{1}{2}(s^2 - s)\left[\frac{1}{4}n^2(n-1)^2 - \frac{1}{2}n(n-1)\right]; \\
\Lambda(K_n +_T G) &= \frac{1}{2}(2+D)\left[\left(\frac{1}{2}n(n+1)s\right)^2 - \frac{1}{2}n(n+1)s\right] - \frac{1}{4}n^2(n+1)^2 W(G) \\
&\quad - \frac{1}{2}n(n-1)s^2\left[2(n-1) + \frac{1}{4}\big(n(n-1)+2\big) + \frac{1}{4}(n-2)(n-3)\right] \\
&\quad - \frac{1}{2}(s^2 - s)\left[\frac{1}{4}n^2(n-1)^2 - \frac{1}{2}n(n-1)\right].
\end{aligned}
$$

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